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The k-Metric Dimension of Double Fan Graph

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ABSTRAK (12 pt)

Diberikan graf terhubung dan sederhana $G = \big(V(G), E(G)\big)$ dan bilangan bulat positif k. Himpunan $S \subseteq V(G)$ disebut sebagai pembangkit k-metrik jika untuk setiap pasang titik berbeda $u, v \in V(G)$, terdapat paling sedikit k titik $w_1, w_2, \dots, w_k \in S$ sedemikian sehingga $d(u, w_i) \neq d(v, w_i)$ untuk setiap $i \in \{1,2,\dots,k\}$, dengan $d(u, v)$ adalah panjang *path* terpendek dari u ke v. Pembangkit k-metrik dengan kardinalitas terkecil disebut basis k-metrik, dan kardinalitas dari basis k-metrik disebut dimensi k-metrik dari graf G yang dinotasikan $dim_k(G)$. Pada tesis ini, akan disajikan nilai dari dimensi k-metrik pada graf *double fan*.

Kata Kunci: pembangkit k-metrik, basis k-metrik, dimensi k-metrik, berdimensi k-metrik, graf double fan

ABSTRACT (12pt)

Given a connected simple graph $G = (V, E)$ and a positive integer k. A set $S \subseteq V$ is said to be a k-metric generator if and only if for any pair of different vertices $u, v \in V$, there exist at least k vertices $w_1, w_2, \dots, w_k \in S$ such that $d(u, w_i) \neq d(v, w_i)$, for every $i \in \{1,2,\dots,k\}$, where $d(u, v)$ is the length of a shortest $u - v$ path. A k-metric generator of minimum cardinality is called k-metric basis and its cardinality, the k-metric dimension of G. In this article we determine the k-metric dimension of double fan and some related double fan graph.

Keywords: k-metric generator, k-metric basis, k-metric dimension, k-metric dimensional, double fan graph.

INTRODUCTION

The concept of k-metric generator was introduced in Estrada-Moreno et al. (2015) as a generalization concept of metric generator. In graph theory, metric generator was previously given by Slater in (Slater, 1988), where the metric generators were called locating set. This characteristic sets were introduced in connection with the problem of uniquely determining the location of an intruder in a network. Several other applications of metric generators have been presented, for example applications of the navigation robots are discussed in (Khuller et al., 1996).

For more realistic settings, k-metric dimension allow to study more general approach of locating problems. Consider some some robots which are navigating, moving from node to node of a network. We assume that robots have communication with a set of landmarks S (a subset of nodes), which provides them the distance to the land marks in order to facilitate the navigation. In this sense, one aim is that each robot is uniquely determined by the landmarks. Suppose that in a specific moment there are two robots x and y whose position are only distinguished by one landmark $s \in S$. If the communication between x and s is unexpectedly blocked, then the robot x will get lost in the sense that it can assume has the position of y. So, for security reasons, we will consider a set of landmarks, where each pair of of nodes distinguished by at least $k \geq 2$ landmarks.

Given a simple and connected graph $G = (V, E)$ we denote by $d(u, v)$ the distance between $u, v \in V$. A set $S \subset V$ is said to be a metric generator for G if for any pair of vertices $x, y \in V$ there exists $s \in S$ such that $d(u, s) \neq d(v, s)$. A minimum metric generator is a metric generator with the smallest possible cardinality among all the metric generators for G. A minimum metric generator is called a metric basis, and its cardinality, the metric dimension of G, denoted by $dim(G)$. Given $S = \{s_1, s_2, \dots, s_l\} \subset V(G)$, we refer to the ordered l-tuple $r(u|S) = \big(d(u, s_1), d(u, s_2), \dots, d(u, s_l)\big)$ as the metric representation of u with respect to S. In this sense, S is a metric generator for G if and only if for every pair of different vertices $u, v \in V$, it follows $r(u|S) \neq r(v|S)$.

Now, in a more general setting, given a positive integer k, a set $S \subseteq V$ is said to be a k-metric generator for G if and only if for any pair of vertices of G distinguished at least k elements of S, in other words, for any pair of different vertices $u, v \in V$, there exists at least k vertices $w_1, w_2, \dots, w_k \in S$ such that

$$d(u, w_i) \neq d(v, w_i), for\ every\ i \in \{1, \dots, k\}.$$

By analogy to the standard case, a k-metric generator of minimum cardinality will be called k-metric basis of G and its cardinality, the k-metric dimension of G, will be denoted by $dim_k(G)$.

If two vertices u, v are adjacent in $G = (V, E)$, then we write $u \sim v$. Given $u \in V$, we define $N_G(u)$ as open neighborhood of u in G, we write $N_G(u) = \{v \in V | u \sim v\}$. The closed neighborhood denoted by $N_G[u]$, equals $N_G(u) \cup \{u\}$. We now recall the join graph $G + H$ of the graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G + H) = V_1 \cup V_2$ and edge set $E(G + H) = E_1 \cup E_2 \cup \{uv | u \in V_1, v \in V_2\}$.

Studies about k-metric dimension of corona product graphs were initiated by (Estrada-Moreno et al., 2015), respectively. In this work we continue with the study of the k-metric dimension of double fan graph.

METHOD

The k-metric dimensional graph

A connected graph G is said to be k-metric dimensional if k is the largest integer such that there exists k-metric basis. Note that every k-metric dimensional graph, for each positive integer $k_1 < k$, we know that G is k_1-metric dimensional. We give a characterization of k-metric dimensional graph obtained in (Estrada-Moreno et al., 2015). We need some terminology. Given two vertices $u, v \in V$, we say that the set of distinctive vertices of u, v is

$$\mathcal{D}_G(u, v) = \{w \in V : d\big(u, w \neq d(v, w)\big)\}.$$

Moreover, the set of non-trivial distinctive vertices of u, v is

$$\mathcal{D}_G^*(u, v) = \mathcal{D}_G(u, v) - \{u, v\}.$$

Theorem 1. *A connected graph G is k-metric dimensional if and only if*

$$k = \min_{u,v \in V(G), u \neq v} |\mathcal{D}_G(u, v)|.$$

Two vertices $u, v \in V$ are called false twins if $N_G(u) = N_G(v)$ and called true twins if $N_G[u] = N_G[v]$. Two vertices u, v are twins if they are true twins or false twins. A vertex u is said to be twin if there exists $v \in V - \{u\}$ such that u and v are twins in G. Notice that two vertices u, v are twins if and only if $\mathcal{D}_G(u, v) = \{u, v\}$.

Corollary 1. *A connected graph G of order $n \geq 2$ is 2-metric dimensional if only if G has twin vertices.*

The k-metric dimensional graph

Once we have presented the results on k-metric dimensional graphs, in this section we give the exact value of k-metric dimension of join graphs. To do so, we need to introduce the necessary terminology and some useful properties like the following lemma.

Lemma 1. *Let G be a connected graph and let $u, v \in V$. If B is a k-metric basis of G and $|\mathcal{D}_G(u, v)| = k$, then $\mathcal{D}_G(u, v) \subseteq B$.*

Given a k-metric dimensional graph G, we define $\mathcal{D}_k(G)$ as the set obtained by the union of the sets of distinctive vertices $\mathcal{D}_G(u, v)$ when $|\mathcal{D}_G(u, v)| = k$. We write

$$\mathcal{D}_k(G) = \bigcup_{|\mathcal{D}_G(u,v)|=k} \mathcal{D}_G(u,v).$$

Corollary 2. *Let G be a k-metric dimensional graph. For any k-metric basis B of a graph G it holds $\mathcal{D}_k(G) \subseteq B$.*

RESULTS AND DISCUSSION

According to Weisstein (Weisstein, 2008), a fan graph $f_{n',n}$ is defined as the graph joint $K_{n'} + P_n$, where $K_{n'}$ is the null graph on n' vertices and P_n is the path graph on n vertices. The case $n' = 1$ corresponds to the usual fan graphs, while $n' = 2$ corresponds to the double fan, and so on. Hence, in this paper, we denote the double fan graph as $f_{2,n}$ and the graph as shown in Figure 1.

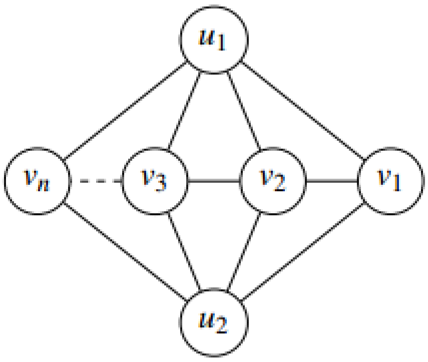

Figure 1. Double Fan Graph

Once we have presented the results on k-metric dimensional graphs, in this section we give the exact value of k-metric dimension of join graphs. To do so, we need to introduce the necessary terminology and some useful properties like the following lemma.

Lemma 2. *Let G be a connected graph and let $u, v \in V$. If B is a k-metric basis of G and $|\mathcal{D}_G(u,v)| = k$, then $\mathcal{D}_G(u,v) \subseteq B$.*

Given a k-metric dimensional graph G, we define $\mathcal{D}_k(G)$ as the set obtained by the union of the sets of distinctive vertices $\mathcal{D}_G(u,v)$ when $|\mathcal{D}_G(u,v)| = k$. We write

$$\mathcal{D}_k(G) = \bigcup_{|\mathcal{D}_G(u,v)|=k} \mathcal{D}_G(u,v).$$

Corollary 3. *Let G be a k-metric dimensional graph. For any k-metric basis B of a graph G it holds $\mathcal{D}_k(G) \subseteq B$.*

Now we determine the k-metric dimension of double fan, double cones, and double fan snake graph.

Lemma 3. *Let $f_{2,n}$ be a double fan graph for $n = 3$ and $n > 4$. If S is 2-metric generator, then* $|S| \geq \left\lceil \frac{n+5}{2} \right\rceil$

Proof. The double fan graph $f_{2,n}$ has $n+2$ vertices. Suppose $S < \left\lceil \frac{n+5}{2} \right\rceil$. We have $\binom{n+2}{|S|}$ possibility for S below.

1. $S = \{u_1, v_1, v_2, \dots, v_k\}$ for $k \in \left\{1, \dots, \left\lceil \frac{n+5}{2} \right\rceil - 2\right\}$.
2. $S = \{u_2, v_1, v_2, \dots, v_k\}$ for $k \in \left\{1, \dots, \left\lceil \frac{n+5}{2} \right\rceil - 2\right\}$.
3. $S = \{u_1, u_2, v_1, v_2, \dots, v_k\}$ for $k \in \left\{1, \dots, \left\lceil \frac{n+5}{2} \right\rceil - 3\right\}$.
4. $S = \{v_1, v_2, \dots, v_k\}$ for $k \in \left\{1, \dots, \left\lceil \frac{n+5}{2} \right\rceil - 1\right\}$.

For every $u, v \in V(f_{2,n})$ and $W \subset S$ such that $|W| = 2$, we have $r(u|W) = r(v|W)$. It contradicts with the fact that S is 2-metric generator, so $|S| \geq \left\lceil \frac{n+5}{2} \right\rceil$.

Theorem 2. *Let $f_{2,n}$ be a double fan graph with $n \geq 2$*

$$dim_2(f_{2,n}) = \begin{cases} 4, & n = 2 \\ 5, & n = 4 \\ \left\lceil \dfrac{n+5}{2} \right\rceil, & n \ otherwise \end{cases}$$

Proof. Its easy to check the 2-metric dimension of $f_{2,2}$ and $f_{2,4}$. We now show the case of $n \geq 4$. We consider two cases according the values of n.

1. For n is odd.

   We choose $S = \{u_1, u_2, v_1, v_3, v_5, \dots, v_n\}$. We have the metric representation for every vertices in $V(f_{2,n})$ with respect to S as follows.

$$\begin{aligned} r(u_1|S) &= (0,2,1,1,\dots,1,1); \\ r(u_2|S) &= (2,0,1,1,\dots,1,1); \\ r(v_1|S) &= (1,1,0,2,\dots,2,2); \\ r(v_2|S) &= (1,1,1,1,\dots,2,2); \\ &\vdots \\ r(v_{n-1}|S) &= (1,1,2,2,\dots,1,1); \\ r(v_n|S) &= (1,1,2,2,\dots,2,0). \end{aligned}$$

   For every $W \subset S$ such that $|W| = 2$, we have $r(u|W) \neq r(v|W)$.

2. For n is even.

   We choose $S = \{u_1, u_2, v_1, v_3, v_5, \dots, v_{n-1}, v_n\}$. We have the metric representation for every vertices in $V(f_{2,n})$ with respect to S as follows.

$$\begin{array}{rcl} r(u_1|S) & = & (0,2,1,1,\dots,1,1); \\ r(u_2|S) & = & (2,0,1,1,\dots,1,1); \\ r(v_1|S) & = & (1,1,1,2,\dots,2,2); \\ r(v_2|S) & = & (1,1,0,1,\dots,2,2); \\ \vdots & & \vdots \\ r(v_{n-1}|S) & = & (1,1,2,2,\dots,0,1); \\ r(v_n|S) & = & (1,1,2,2,\dots,2,0). \end{array}$$

   For every $W \subset S$ such that $|W| = 2$, we have $r(u|W) \neq r(v|W)$.

We have S is 2-metric generator. According to Lemma 3, S is 2-metric basis. Thus, $dim_2(f_{2,n}) = \left\lceil \frac{n+5}{2} \right\rceil$.

CONCLUSION

In this work we have that double fan graph $f_{2,n}$ for $n = 3$ and $n > 4$. If S is 2-metric generator, then $|S| \geq \lceil \frac{n+5}{2} \rceil$ and we have the metric dimension of double fan $f_{2,n}$ for $n \geq 2$, then

$$dim_2(f_{2,n}) = \begin{cases} 4, & n = 2; \\ 5, & n = 4; \\ \left\lceil \dfrac{n+5}{2} \right\rceil, & n\ otherwise. \end{cases}$$

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